

# Summary of *Self-similar Markov trees and scaling limits*

Jean Bertoin, Nicolas Curien, and Armand Riera (arXiv:2407.07888, 2024)

The book *Self-similar Markov trees and scaling limits* –to be published by Cambridge University Press– by Jean Bertoin, Nicolas Curien, and Armand Riera presents a unifying and highly general theory of random continuum trees constructed through the interplay of Markov branching structures and self-similarity. It establishes a framework capable of encompassing, as particular cases, nearly all classical continuum random trees known so far, including Aldous’ Brownian Continuum Random Tree [1], the stable Lévy trees of Duquesne and Le Gall [4], and more recent models such as growth-fragmentation and fragmentation trees as well as random trees arising in models of planar geometry such as the Brownian sphere [2, 6]. The authors’ purpose is not to restate known convergence results, but to build a single coherent probabilistic object –*self-similar Markov trees* (ssMt)– that acts as a universal scaling limit for a broad class of discrete random trees sharing a Markovian branching mechanism and possessing a self-similar scaling property.

The starting point of the construction is the observation that many random trees appearing in probability theory and statistical physics possess a recursive branching property: subtrees above a given height or generation are conditionally independent and their distributions depend only on a size parameter associated with the branching node. At the same time, when properly rescaled, these subtrees resemble the whole tree, leading to a fractal-like self-similar structure. The authors formalize this intuition by defining a random compact rooted real tree  $(\mathcal{T}, d)$  and an auxiliary function  $g : \mathcal{T} \rightarrow \mathbb{R}_+$ , called a decoration, which may represent the “type” attached to each point. The triplet  $(\mathcal{T}, d, g)$  is called a self-similar Markov tree if it satisfies two axioms: a Markov branching property stating that the subtrees above a given height are independent conditionally on their decoration and a self-similarity property expressing that the law of a rescaled subtree depends only on its decoration through a deterministic scaling relation.

More conceptually, the construction of self-similar Markov trees can be viewed as arising from performing a Lamperti transformation on the *branching Lévy processes* introduced and studied by Bertoin and Mallein [3]. Roughly speaking, a branching Lévy process is a system of particles evolving on the real line, where each particle moves according to an independent Lévy process and, at random times governed by the jumps of this process, gives birth to new independent copies of itself shifted in space. Applying a Lamperti-type time change and passing to the exponential to this branching structure transforms the linear spatial motion into a multiplicative scaling of distances, and turns the additive genealogical structure of the branching Lévy process into a continuous real tree that inherits the same self-similar Markov property. In this way, the ssMt construction may be interpreted as the branching analog of the famous Lamperti construction of positive self-similar Markov processes from Lévy processes. In particular a ssMt is characterized by a quadruplet  $(a, \sigma^2, \mathbf{\Lambda}; \alpha)$  where  $\alpha > 0$  is the self-similarity parameter,  $a \in \mathbb{R}$ ,  $\sigma^2 \geq 0$  and  $\mathbf{\Lambda}$  is the branching generalization of the Lévy measure.

A central analytical object in this framework is the *cumulant function*, which encapsulates the infinitesimal branching behavior of the underlying Lévy process and governs many large-scale features of the self-similar Markov tree. The cumulant function serves simultaneously as

the Laplace exponent of the associated additive martingale and as the key to determining the scaling exponents that appear in the limit theorems. On the resulting tree, it enables us to define natural measures, notably the length measure on the skeleton and the harmonic measure on the leaves. These measures then serve to establish spinal decompositions, a central tool in branching structures.

The authors proceed then to formulate invariance principles that rigorously connect discrete and continuous worlds. They consider sequences of discrete Markov branching trees, for example multi-type Galton-Watson trees, fragmentation trees [5], or trees extracted from combinatorial structures such as planar maps. Each discrete model is assumed to have a branching kernel that depends only on a size or type parameter and to satisfy certain criticality and scaling assumptions. Such trees are then shown to converge in the scaling limits in a strong Gromov-type topology, toward a limiting ssMt. The theory naturally reproduces and extends many classical results. When the discrete trees are critical Galton-Watson trees with finite variance offspring, the scaling limit is the Brownian Continuum Random Tree of Aldous. When the offspring distribution has a heavy tail with index  $\alpha \in (1, 2)$ , the limit is a stable Lévy tree as studied by Duquesne and Le Gall. Other choices of branching rules yield fragmentation or growth-fragmentation trees, which appear in models of random planar maps or parking model on random trees.

Beyond the precise probabilistic theorems, the book offers a conceptual unification of ideas scattered across many recent papers in the field. By showing that a vast family of random trees converge toward self-similar Markov trees, Bertoin, Curien, and Riera not only recover all previously known scaling limits but also predict new ones that may arise in unexplored discrete models. Their approach demonstrates that many complex tree-like random structures can be understood as manifestations of a single universal probabilistic mechanism governed by self-similarity and Markov independence. The book, extending over two hundred pages and illustrated with numerous figures, represents a synthesis of several decades of research on random trees and an ambitious attempt to build a general theory of self-similar random geometry.

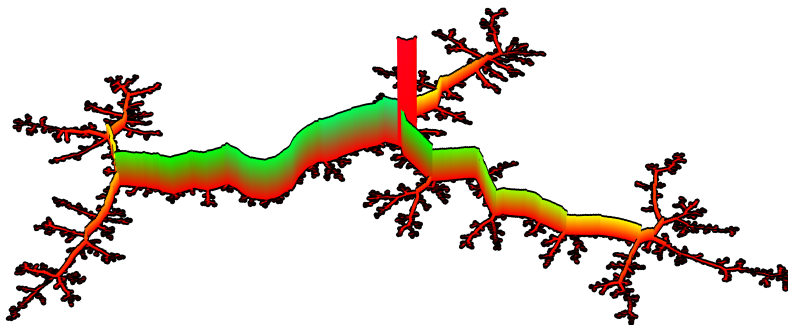


Figure 1: Illustration of a self-similar Markov tree (embedded in the plane  $\mathbb{R}^2$ ) where its decoration function is represented in the third (vertical) dimension.

## References

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