The Weierstrass Drift: A Toy SLE Brownian Model

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We introduce the Weierstrass Drift as a deterministic toy model for the Schramm–Loewner evolution process (SLE) driven by Brownian motion. More precisely, given $T > 0$, we are interested in the solution $g : [0,T] \times \mathbb{H} \to \mathbb{C}$ of the chordal backward Loewner (partial) differential equation (in short, Loewner equation),

$$\forall\,(t,z) \in [0,T] \times \mathbb{H} : \quad \frac{\partial g}{\partial t} = -\frac{2}{g(t,z) - c\,\mathcal{W}(t)} \qquad , \qquad (\mathcal{L}_{\mathcal{W}})$$

where $\mathbb{H} = \{z = x + i\,y\,,\, \mathcal{I}m(z) = y > 0\}$ denotes the upper half-plane, $\mathcal{W}$ the Weierstrass function $x \mapsto\mapsto \mathcal{W}(x) = \sum_{n=0}^{\infty} \lambda^n \cos\left(2\,\pi\,b^n\,x\right)$, with $\lambda \in]0,1[$, $N_b \geqslant 2$, $c > 0[$, with the initial condition $g(t_0, z) = z$.

Given $t \in [0,T]$, we define *the hull* K_t as the two-dimensional domain such that

$$K_t = \left\{z \in \overline{\mathbb{H}}\,,\, g(s,z) = \mathcal{W}(s) \quad \text{for some } s \leqslant t\right\}. \qquad (\mathcal{R}\,1)$$

The now classical condition obtained by Joan Lind and Jessica Robbins in [Lin05] ensures that, if the Lip $\left(\frac{1}{2}\right)$ norm of the drift is below 4, the resulting SLE trace is a simple curve.

We hereafter revisit this condition, by relying on our previous works [DL25d], [DL24c], [DL24a], [DL25a], [DL24b], [DL25b], [DL25c], and show that not only we can obtain more general results, namely, for $\delta \in \left]0, \frac{1}{2}\right]$, bounds for the Lip (δ) norm of the Weierstrass function, but, also, for $\delta = \frac{1}{2}$, significantly improve the bounds obtained by J. Lind and J. Robbins in [LR17]. Our results are all the more important, insofar as they enable us to more precisely determine the range of values of the real parameter c such that the Weierstrass drift shows a phase transition (corresponding to traces of SLE_κ as simple or multiple curves). We also obtain the results associated with the less restrictive general case of a Lip (δ) drift, with $0 < \delta \leqslant \frac{1}{2}$, a configuration all the more interesting, insofar as the Brownian motion is Hölder continuous with associated Hölder exponent $\delta \in \left]0, \frac{1}{2}\right[$. Our general Weierstrass toy model could, therefore, provide a better understanding of the singularities of multifractal SLE traces.

We go much further than what is done in [LR17], since we conduct a comparative study, in the case of Weierstrass-prefractal driven hulls – when the drift is a (piecewise linear) Weierstrass-prefractal function, corresponding to a realistic approximation of the Weierstrass function, which enables us to take into account its roughness – and in the case of Weierstrass-truncated driven hulls – when the drift is a sharp truncation of the Weierstrass function, by nature a smooth function, as is done in [LR17]. By computing the associated Chhabra–Jensen multifractal spectra, we are finally able to show that the polygonal depth of the drift controls the

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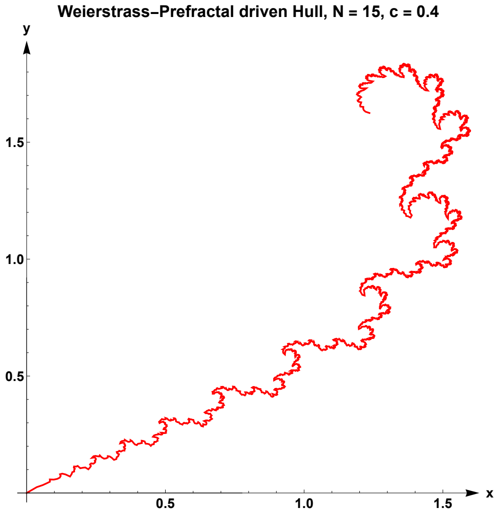
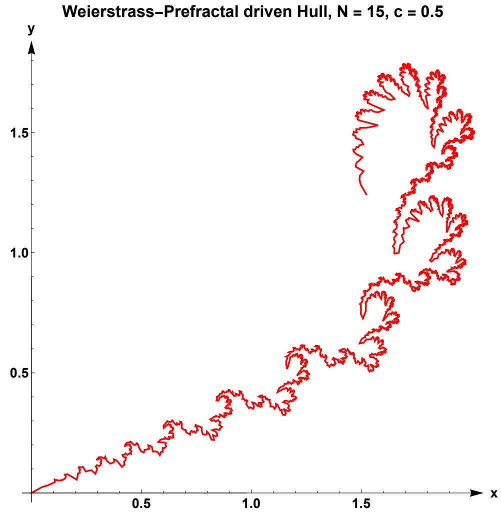
appearance of new scales both in the drift and in the resulting hull, which highlights a fundamental connection between the regularity of the drift and the geometric complexity of the trace: small oscillations of the drift lead to oscillations of the hull.

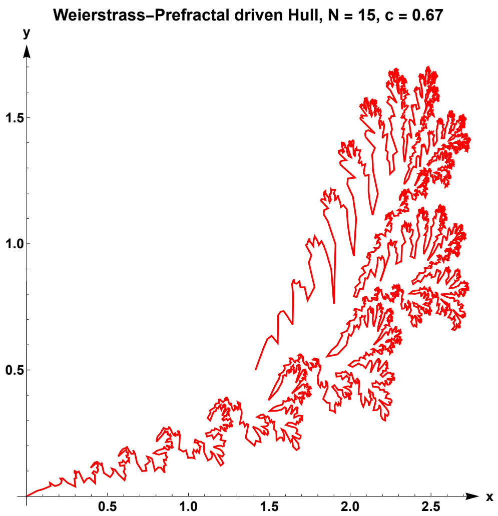
We therefore provide numerical evidence of the fact that the Weierstrass drift mimics key features of Brownian driving terms – such as the roughness, the box-counting dimensions of the resulting hulls, and *quasi self-similarity* (or self-shape similarity; see [DL25d]). This suggests that it can serve as a tractable toy model for the multifractality and phase transition study of SLE traces.



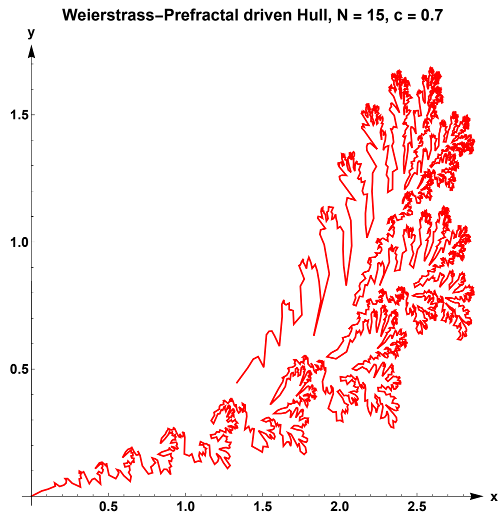
(a) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.4$.**



(b) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.5$.**



(c) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.67$.**



(d) **The Weierstrass prefractal driven hull, for $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$, $c = 0.7$.**

Figure 1: **The phase transition, in the case of Weierstrass-prefractal driven hulls, when $\lambda = \frac{1}{\sqrt{2}}$, $N_b = 2$, $N = 15$.**

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